

Quantum Parrondo's games constructed by quantum random walks

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We construct a Parrondo's game using discrete time quantum walks. Two losing games are represented by two different coin operators. By mixing the two coin operators $U_A(\alpha_A, \beta_A, \gamma_A)$ and $U_B(\alpha_B, \beta_B, \gamma_B)$, we may win the game. Here we mix the two games in position instead of time. With a number of selections of the parameters, we can win the game with sequences ABB, ABBB, *et al.* If we set $\beta_A = 45^\circ, \gamma_A = 0, \alpha_B = 0, \beta_B = 88^\circ$, we find the game 1 with $U_A^S = U^S(-51^\circ, 45^\circ, 0)$, $U_B^S = U^S(0, 88^\circ, -16^\circ)$ will win and get the most profit. If we set $\alpha_A = 0, \beta_A = 45^\circ, \alpha_B = 0, \beta_B = 88^\circ$ and the game 2 with $U_A^S = U^S(0, 45^\circ, -51^\circ)$, $U_B^S = U^S(0, 88^\circ, -67^\circ)$, will win most. And game 1 is equivalent to the game 2 with the changes of sequences and steps. But at a large enough steps, the game will loss at last.

I. INTRODUCTION

Parrondo's games present an apparently paradoxical situation where individually losing games can be combined to win [1–3]. Parrondo's games have important applications in many physical and biological systems. For example in control theory, the analogy of Parrondo's games can be used to design a second order switched mode circuit which is unstable in either mode but is stable when switched [4]. Recently, quantum version of Parrondo's games was also introduced in [5–9].

Quantum walks(QWs), as the quantum version of the classical random walks, were first introduced in 1993 [10] (For overviews, see [11, 12]). According to the time evolution, QWs can be divided into discrete-time (DTQWs) and continuous-time [13] QWs (CTQWs). A number of quantum algorithms based on QWs have already been proposed in [14–19], and QWs are found to be universal for quantum computation [20, 21].

In this paper, we construct a Parrondo's games using QWs. Two models fo Parrondo's game using QWs were introduced in [6, 9], but both of them are switched according to the time. Here we construct the games with coin operators alternated depending on the position in one step.

II. DISCRETE TIME QUANTUM WALKS ON A LINE

In this paper, we concern with the discrete-time quantum walks. To be consistent, we adopt analogous definitions and notations as those outlined in [22]. The total Hilbert space is given by $\mathcal{H} \equiv \mathcal{H}_P \otimes \mathcal{H}_C$, where \mathcal{H}_P is spanned by the orthonormal vectors $\{|x\rangle : x \in \mathbb{Z}\}$ which represents the position of the walker and \mathcal{H}_C is the Hilbert space of chirality, or “coin” states, spanned by the orthonormal basis $\{|L\rangle, |R\rangle\}$.

Each step of the quantum walk can be split into two operations: the flip of a coin and the position motion of the walker according to the coin state.

Here, for simplicity, we choose a Hadamard coin as the normal quantum walk's coin, so the coin operator can be written as

$$\hat{H} |\downarrow\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle + |\uparrow\rangle), \quad \hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (1)$$

The position displacement operator is given by

$$\hat{S} = e^{i\hat{p}\hat{\sigma}_z} = \sum_x \hat{S}_x, \quad (2)$$

where \hat{p} is the momentum operator, $\hat{\sigma}_z$ is the Pauli- z operator,

$$\hat{S}_x = |x+1\rangle\langle x| \otimes |\uparrow_x\rangle\langle\uparrow_x| + |x-1\rangle\langle x| \otimes |\downarrow_x\rangle\langle\downarrow_x|. \quad (3)$$

Therefore, the state of the walker after N steps is given by

$$\begin{aligned} |\Psi_N\rangle &= [\hat{S}(\hat{I}_P \otimes \hat{H}_C)]^N |\Psi_0\rangle \\ &= \left[\sum_x \hat{S}_x(\hat{I}_P \otimes \hat{H}_C) \right]^N |\Psi_0\rangle, \end{aligned} \quad (4)$$

where $|\Psi_0\rangle$ is the initial state of the system.

For generalized DTQWs, we use a $U(2)$ matrix:

$$U_{\alpha,\beta,\gamma,\theta} = e^{i\theta} \begin{pmatrix} e^{i\alpha} \cos \beta & -e^{-i\gamma} \sin \beta \\ e^{i\gamma} \sin \beta & e^{-i\alpha} \cos \beta \end{pmatrix} \quad (5)$$

instead of the Hadamard operator in Eq. (4). We can easily know the QWs using a $SU(2)$ operator:

$$U_{\alpha,\beta,\gamma}^S = \begin{pmatrix} e^{i\alpha} \cos \beta & -e^{-i\gamma} \sin \beta \\ e^{i\gamma} \sin \beta & e^{-i\alpha} \cos \beta \end{pmatrix}, \quad (6)$$

have the same properties as using a $U(2)$ coin operator[24]. In this paper we always use the $SU(2)$ operator.

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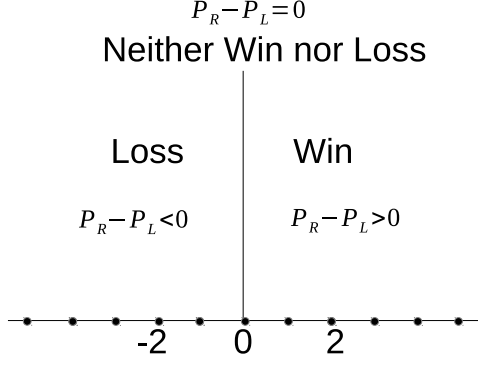


Figure 1. Pictorial illustration of the conditions for win or loss for QWs on a line.

III. PARRONDO'S GAME USING QRW WITH POSITION DEPENDENT ON COIN OPERATOR

Parrondo's games arise in the following situation when we have two games that are losing when played separately, but the two games played in combination will form an overall winning game.

Here we present a scheme of a player playing games using DTQW. He has two games A and B, but he does not play the two games alternated according to the time, but the position [23]. Alternating the coin operation step-by-step has been discussed in [9]. Here we discuss Parrondo's games using QRWs with the coin operation alternated with the position in one step.

The game is constructed as follows:

- Both game A and B are represented by different quantum operators $U(\alpha_A, \beta_A, \gamma_A)$ and $U(\alpha_B, \beta_B, \gamma_B)$.
- The state is in $|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle(|\downarrow\rangle + i|\uparrow\rangle))$ initially.
- Game A and B are played alternately in different positions in one step, instead of step by step. i.e. game A is played on site $x = nq$ and game B is played on site $x \neq nq$. The evolution operator can be written as:

$$U = \sum_{x=nq, n \in \mathbb{Z}} \hat{S}_x U(\alpha_A, \beta_A, \gamma_A) + \sum_{x \neq nq, n \in \mathbb{Z}} \hat{S}_x U(\alpha_B, \beta_B, \gamma_B), \quad (7)$$

where q is the period, n is an integer, and the final state after N steps is given by

$$|\Psi_N\rangle = U^N |\Psi_0\rangle. \quad (8)$$

For $q = 3$, it means we play games with the sequence ABBABB on the line.

As denoted in Fig. 1, after N steps, if the probability P_R of the walker to be found in the right of the origin, is

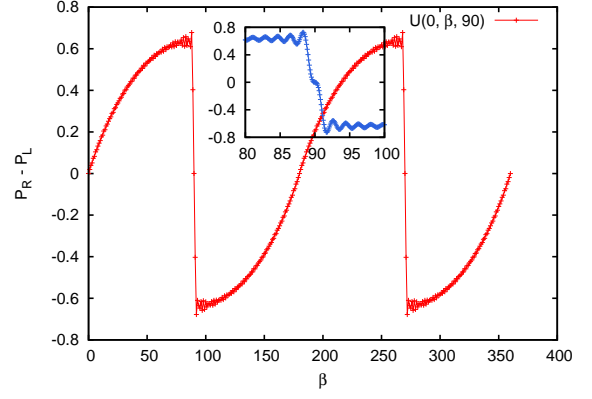


Figure 2. (Color online) $P_R - P_L$ of the walker after $t = 100$ steps QWs for the initial state $|\Psi_0\rangle = 1/\sqrt{2}(|0L\rangle + i|0R\rangle)$, and the coin operator $U^S = U^S(0, \beta, 90)$. The figure inside shows $P_R - P_L$ vary with parameter $\beta \in [80, 100]$.

greater than the probability P_L in the left of the origin, that is $P_R - P_L > 0$, we consider the player win $P_R - P_L$. Similarly, if $P_R - P_L < 0$, the player losses $P_L - P_R$. If $P_R - P_L = 0$, it means the player neither losses nor wins.

IV. RESULTS

Here we use $P_R - P_L = \sum_{x>0} (P(x) - P(-x))$, where $P(x)$ is the probability of the particle to be found at x , but not the average position $\langle x \rangle$ to indicate the player win or loss, because sometimes $\langle x \rangle = \sum_{x>0} x(P(x) - P(-x))$ may be positive, but $P_R - P_L$ be negative, and vice versa. So we can not use $\langle x \rangle$ to indicate the player win or loss.

Theorem 1. *If the initial state $|\Psi_0\rangle = 1/\sqrt{2}(|0L\rangle + i|0R\rangle)$, after t steps quantum walk: $P_R - P_L = M(\beta, t) \sin(\alpha + \gamma)$, where $M(\beta, t)$ only depends on β and t .*

Proof. The same as the proof of $\langle x \rangle = G(\beta, t) \sin(\alpha + \gamma)$ from [24], we can easily get $P_R - P_L = M(\beta, t) \sin(\alpha + \gamma)$. \square

Theorem 1 shows $P_R - P_L = M(\beta, t) \sin(\alpha + \gamma)$, then if we set $\alpha + \gamma = \pi/2$, we can calculate $M(\beta, t)$ varied with the change of β . Fig. 2 shows the $P_R - P_L$ varied with the change of β , after 100 steps QWs with the initial state $|\Psi_0\rangle = 1/\sqrt{2}(|0L\rangle + i|0R\rangle)$, and the coin operator $U^S = U^S(0, \beta, 90)$ (Here 90 denote 90 degree, in the following, we always use this setting). The figure inside shows $P_R - P_L$ varies with the parameter $\beta \in [80, 100]$. From the figure, we can know that when $\beta \approx 88$, $P_R - P_L$ gets its maximum value. In the following, we set $\beta_A = 45, \beta_B = 88$.

Fig. 3 shows the $P_R - P_L$ of the walker after 100 steps QWs with $q = 3$ (a sequence of games ABB), and $U_A^S = U^S(15, 45, 30)$, $U_B^S = U^S(\alpha_B, 88, 0)$. From the figure, we

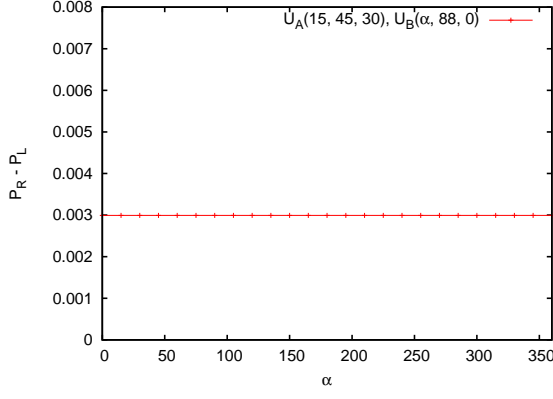


Figure 3. (Color online) $P_R - P_L$ of the walker after 100 steps QWs for the initial state $|\Psi_0\rangle = 1/\sqrt{2}(|0L\rangle + i|0R\rangle)$, with $q = 3$ (for a sequence of games ABB), and $U_A^S = U^S(15, 45, 30)$, $U_B^S = U^S(\alpha_B, 88, 0)$.

can know that $P_R - P_L$ does not vary with the change of α_B . So in the following, we always calculate with $\alpha_B = 0$.

Fig. 4 shows $P_R - P_L$ of the walker after 100 steps QWs for the initial state $|\Psi_0\rangle = 1/\sqrt{2}(|0L\rangle + i|0R\rangle)$, with $q = 3$, and $U_A^S = U^S(\alpha_A, 45, 0)$, $U_B^S = U^S(0, 88, \gamma_B)$, where $\alpha_A, \gamma_B \in [-180, 0]$. From Theorem 1, we can know that after 100 steps QWs with any $\alpha_A, \gamma_B \in [-180, 0]$, game A and game B are losing when played separately, but Fig. 4 shows that there are many choices of α_A and γ_B that can win the game (i.e. $P_R - P_L > 0$) with sequence of games ABB , i.e. the Parrondo's paradox arises in quantum version. The maximum of $P_R - P_L \approx 0.00673$, when $U_A^S = U^S(-51, 45, 0)$, $U_B^S = U^S(0, 88, -16)$.

The same as above, Fig. 5 shows $P_R - P_L$ of the walker after 100 steps QWs with a sequence of games ABB and $U_A^S = U^S(0, 45, \gamma_A)$, $U_B^S = U^S(0, 88, \gamma_B)$, where $\gamma_A, \gamma_B \in [-180, 0]$. $\gamma_A, \gamma_B \in [-180, 0]$. Game A and game B are losing when played separately, but combining them as ABB , we will win with many choices of γ_A and γ_B . The maximum of $P_R - P_L \approx 0.00673$, when $U_A^S = U^S(0, 45, -51)$, $U_B^S = U^S(0, 88, -67)$.

Fig. IV shows the above two games of the maximum $P_R - P_L$, game 1: with $U_{A1}^S = U^S(-51, 45, 0)$, $U_{B1}^S = U^S(0, 88, -16)$; games 2 with $U_{A2}^S = U^S(0, 45, -51)$, $U_{B2}^S = U^S(0, 88, -67)$. From the left figure, we know that game 1 and game 2 have the same result of win or loss with different steps. The right figure shows that game 1 and game 2 will win in the same way with the increasing of q . So game 1 is equivalent to game 2, then in the following, we only need to study the game 1.

Next, we want to know for the sequence of game ABB whether there still exists the effect of Parrondo's paradox with the increasing of steps. First, we need to know games A with $U_A^S = U^S(-51, 45, 0)$ and B with $U_B^S = U^S(0, 88, -16)$ still loss after different steps QWs? Fig. 7 shows that the games A and B still loss after different steps QWs, and the $P_R - P_L$ decreases fast and will tend to stable with the increasing of steps. Second, Fig. 8

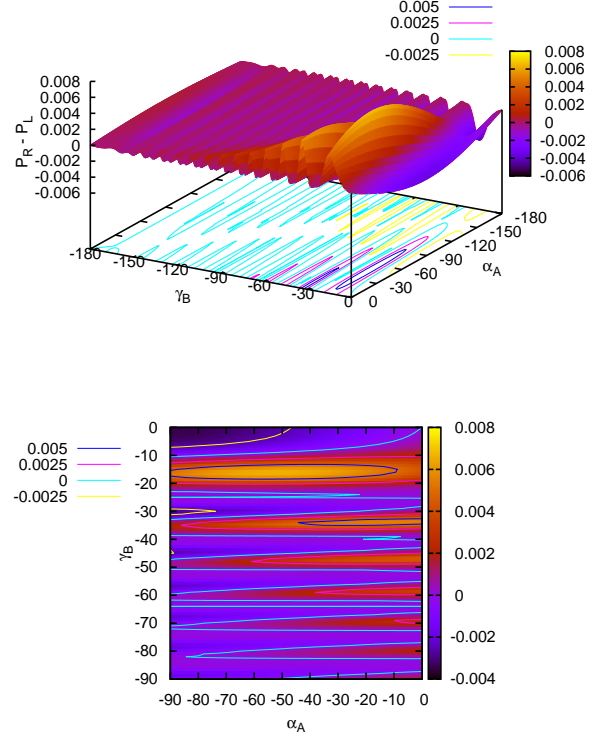


Figure 4. (Color online) $P_R - P_L$ of the walker after 100 steps QWs for the initial state $|\Psi_0\rangle = 1/\sqrt{2}(|0L\rangle + i|0R\rangle)$, with $q = 3$ (for a sequence ABB), and $U_A^S = U^S(\alpha_A, 45, 0)$, $U_B^S = U^S(0, 88, \gamma_B)$, where $\alpha_A, \gamma_B \in [-180, 0]$. The bottom of the left figure is the contour line of $P_R - P_L$. The right figure shows the contour line of $P_R - P_L$, when $\alpha_A, \gamma_B \in [-90, 0]$.

shows the combined game in the situation of sequence ABB situation, with the increasing of steps (only even steps), the result of the game will always fluctuate, and will loss with a large enough step.

V. CONCLUSION

In this paper, we have constructed Parrondo's games by using the one-dimensional discrete time quantum walks. The game is constructed by two losing games A and B with two different biased coin operators $U_A(\alpha_A, \beta_A, \gamma_A)$ and $U_B(\alpha_B, \beta_B, \gamma_B)$, but different from the time dependent sequences of games in [9], here we consider the position dependent sequences of games. With a number of selections of $\alpha_A, \beta_A, \gamma_A, \alpha_B, \beta_B, \gamma_B$, we can form a winning game with sequences ABB , $ABBB$, *et al.* If we set $\beta_A = 45, \gamma_A = 0, \alpha_B = 0, \beta_B = 88$, we find the *game 1* with $U_A^S = U^S(-51, 45, 0)$, $U_B^S = U^S(0, 88, -16)$ will win most. If we set $\alpha_A = 0, \beta_A = 45, \alpha_B = 0, \beta_B = 88$, the *game 2* with $U_A^S = U^S(0, 45, -51)$, $U_B^S = U^S(0, 88, -67)$ will win most. And *game 1* is equivalent to the *game 2* with the changes of

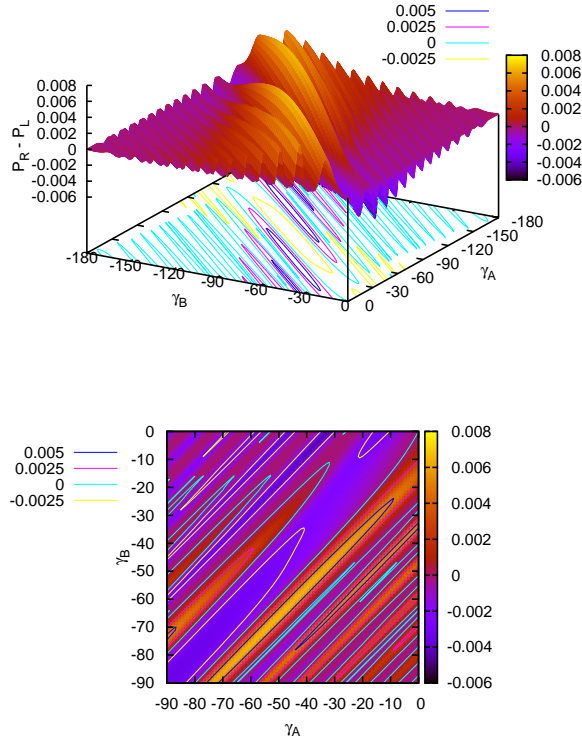


Figure 5. (Color online) $P_R - P_L$ of the walker after 100 steps QWs for the initial state $|\Psi_0\rangle = 1/\sqrt{2}(|0L\rangle + i|0R\rangle)$, with $q = 3$ (for a sequence ABB), and $U_A^S = U^S(0, 45, \gamma_A)$, $U_B^S = U^S(0, 88, \gamma_B)$, where $\gamma_A, \gamma_B \in [-180, 0]$. The bottom of the left figure is the contour line of $P_R - P_L$. The right figure shows the contour line of $P_R - P_L$, when $\gamma_A, \gamma_B \in [-90, 0]$.

sequences and steps. But at a large enough steps, the game will loss.

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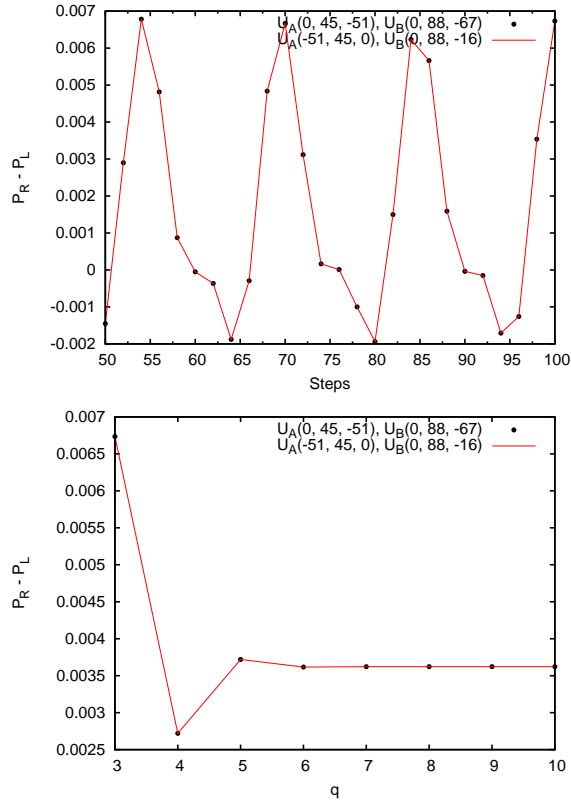


Figure 6. (Color online) Left figure: $P_R - P_L$ of the walker after QWs different steps with $q = 3$. Right figure: $P_R - P_L$ of the walker after 100 steps QWs with different q (sequences of games, e. x. $q = 4$, $ABBB$). 100 steps with (red line) $U_A^S = U^S(-51, 45, 0)$ and $U_B^S = U^S(0, 88, -16)$ or (black point) $U_A^S = U^S(0, 45, -51)$ and $U_B^S = U^S(0, 88, -67)$.

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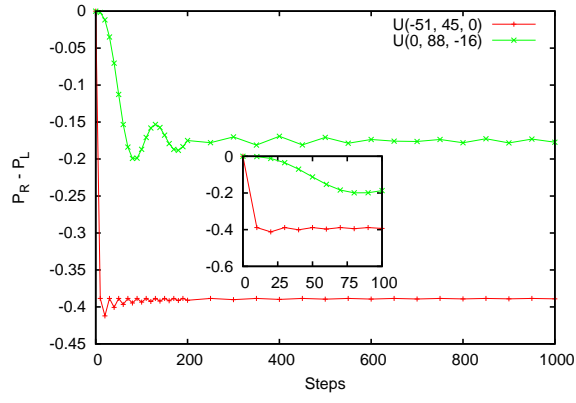


Figure 7. (Color online) $P_R - P_L$ of the walker for QWs after t steps, with initial state $|\Psi_0\rangle = 1/\sqrt{2}(|0L\rangle + i|0R\rangle)$, and coin operator $U^S(-51, 45, 0)$ (red line) or $U^S(0, 88, -16)$ (green line).

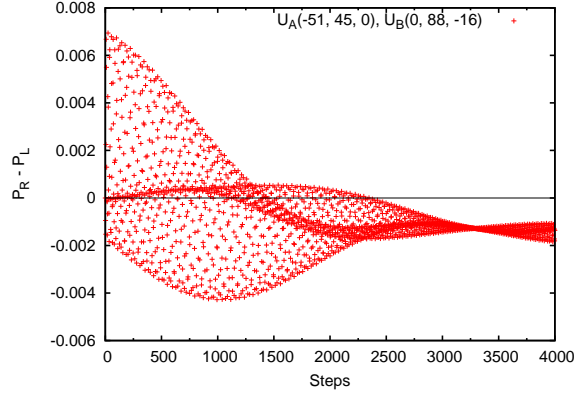


Figure 8. (Color online) $P_R - P_L$ of the walker after QWs different steps with $q = 3$, $U_A^S = U^S(-51, 45, 0)$, $U_B^S = U^S(0, 88, -16)$. (only even steps)